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# Method for analytically calculating BER (bit error rate) in presence of non-linearity

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## Abstract

Methods for calculating bit error rate (BER) using probability density function (PDF) of impairments (ISI, crosstalk, jitter etc.) have been studied and presented in the past. These approaches calculate PDF of the signal & impairments at the decision point based on linear system analysis and deduce the BER from tail probability. A typical high speed system also has significant non linearity in the signal path. We present a method to modify the probability distribution function to account for non-linearity in the path. Once the PDF is correctly modified, the tail probability methods to determine BER can be applied. This way we can combine any linear system(s) with a non-linear system(s) and predict its BER based on knowledge of channel and impairments. Static, memory-less non-linearity polynomials are considered in this paper. The methodology can also be adapted for time varying, frequency dependent non linearity.

## Author(s) Biography

**Gaurav Malhotra** received his M.S. degree in EE from Pennsylvania State University in 2004. His focus areas are communications and signal processing, he has been working on system architecture and algorithms for high speed communication links for 8 years. He currently works in the SerDes architecture group at Xilinx; previously he has worked on 10G-BaseT transceiver architecture.

## Introduction

First, we will review the BER calculation methodology based on linear system modeling. This methodology is very well developed in literature and used in the industry [Several references are provided at the end]. This approach involves calculating the PDF of useful signal and impairments (both voltage and timing) at the decision point. Voltage impairments (ISI / Crosstalk etc.) are typically calculated using pulse response (or impulse response) which can be obtained from S-parameters and reflection coefficient measurements. PDF of all impairments are convolved to obtain the final PDF. In a similar manner, PDF of timing impairments (RJ, PSIJ, SJ etc.) can be calculated. Conditional probability methods are then applied to calculate joint PDF. Once the overall joint PDF is available, tail probability can be easily calculated to give BER.

Second, we will review a common model of non-linearity in a system which concatenates a linear system with a polynomial (typically Taylor series). This methodology is used to model static time invariant non linearity, and for a lot of high speed links is a reasonably accurate model of actual non linearity. Another more accurate method of modeling frequency dependent non linearity is by using Volterra series.

We will then present a method to accurately modify the joint PDF in presence of nonlinearity, present BER results for a typical high speed link. We will compare BER results for NRZ and PAM4 modulation – with and without nonlinearity. Although the methodology of this work can be extended to modeling frequency dependent non linearity, the results presented in this paper are directly applicable to static non linearity only.

Since some impairments such as crosstalk are inherently non-white, the instantaneous error rate will be different from average error rate. To keep the focus on methodology for handling non linearity while calculating BER, we model non-white impairments such as crosstalk as colored noise and use the exact PDFs, but report the average BER.

## LTI systems: BER methodology overview

In this section we review the methodology to model an LTI (linear time invariant) system and calculate the overall joint PDF therefrom. We will also set notations used in rest of the paper. Once the overall joint PDF is available, tail probability can be easily calculated to give BER. In the following section, we will discuss as to how we can modify this methodology to accurately account for non-linearity in the path.

Analog waveform at the input to the decision slicer can be modeled as:

$$x(t+\phi) = \mathrm{Signal}(t+\phi) + Crosstalk(t+\phi) + n(t+\phi)$$

Where $Signal(t)$ represents the useful signal of interest, $Crosstalk(t)$ represents crosstalk from adjacent transmitters, $n(t)$ represents white Gaussian noise. Crosstalk is a colored (correlated) noise source whereas *n(t)* is white. Variable $\phi$ represent timing jitter (this can be a combination of various jitter sources like RJ, PSIJ, and SJ etc.) These impairments are depicted in the link model of Figure 1.

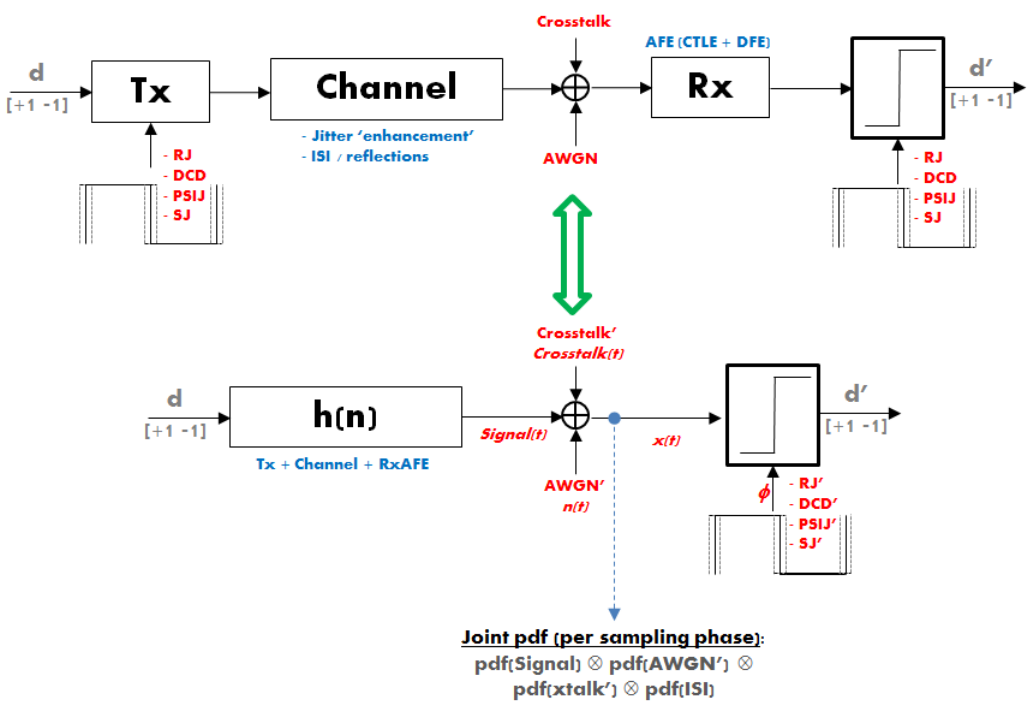


**Figure 1**

To reach our final goal of determining the PDF of x(t) from which we can deduce error probability, we need to calculate the individual PDFs of signal, crosstalk, AWGN at the decision slicer input. These individual PDFs can then be convolved to get the final joint PDF of x(t). To get the distribution of various components, we first express waveform x(t) in form of pulse response of main signal path, pulse response of crosstalk path as:

$$x(t+\ \phi) = \sum_k d_k\, S(t - kT +\ \phi) + \sum_k d'_k\, C(t - kT +\ \phi) + n(t+\ \phi)$$

Where {d} represent the transmit alphabet (typically +1, -1 for NRZ signaling). $d_k$ is assumed to be an i.i.d (identical, independently distributed) sequence. Pulse response is easily derived from impulse response which can be obtained from S-parameters and reflection coefficients. Note that the above equation can be modified for multiple crosstalk channels or other impairment by adding additional independent variables (convolving additional PDFs).

We typically want to find the BER at various sampling phases within one symbol period (also called unit interval/ UI). Denoting one symbol period as T, the sampled signal can be represented as:

$$x(\mathrm{mT} +\ \Delta +\ \phi) = \sum_k d_k\, S(\mathrm{mT} +\ \Delta - kT +\ \phi) + \sum_k d'_k\, C(\mathrm{mT} +\ \Delta - kT +\ \phi) + n(\mathrm{mT} +\ \Delta +\ \phi)$$

Where m denotes the time index in terms of UI and the variable Δ denotes sampling phase. Δ can vary from 0:2Π.

Since PDF of sum of variables is the convolution of individual PDFs, the PDF of x(t) can be written as:

$F_X(x) \quad = \qquad F_S(\text{signal}) \otimes F_C(\text{Crosstalk}) \otimes F_N(\text{AWGN})$

Where ⊗ denotes convolution & $F_X(x)$ denotes the PDF of variable X. Note that the PDFs do not necessarily need to have a closed form solution. For example, PDF of signal at the channel output can be computed using {d} & the channel impulse response. This two dimensional quantity (x-axis: value/voltage & y-axis: probability of occurrence of the x-axis value) can be stored in any desired precision. Error probability (BER) for the variable X can then be written as:

$$P(error) = \sum_k P(error|d_k)P(d_k)$$

Where $P(error|d_k)$ denotes the probability of error when $d_k$ was transmitted. For NRZ signaling, where {d} = [1 -1] and IID, the above equation simplifies to:

$$P(error) = P(error|1) * 0.5 + P(error|-1) * 0.5$$

$P(error|1)$ can be derived from the PDF of X as: $P(error|1) = \int_{-\infty}^{SS} F_X(x|1)$

SS denotes the Slicer sensitivity (latch sensitivity). If signal magnitude is less than SS, we declare that an error event has happened. Thus, $P(error|d_k)$ can be derived for all $d_k$ and thus the overall BER can be calculated.

So far we have not included timing jitter in our equations. To account for the effect of timing jitter, we can calculate the above PDF $F_X(x)$ for each sampling phase Δ and condition each PDF on the probability of occurrence of each corresponding phase. Let $BER_{\Delta k}$ denote the error probability when sampling phase is $\Delta_k$. Let $F_\Delta$ denote the distribution of sampling phase Δ. The BER taking timing jitter into account can then be computed as:

$$BER = \sum_k BER_{\Delta k} \quad F\Delta(k)$$

If the above variables can be represented in closed form, then the summation will change to integral. The above methodology is pictorially denoted in Figure 2

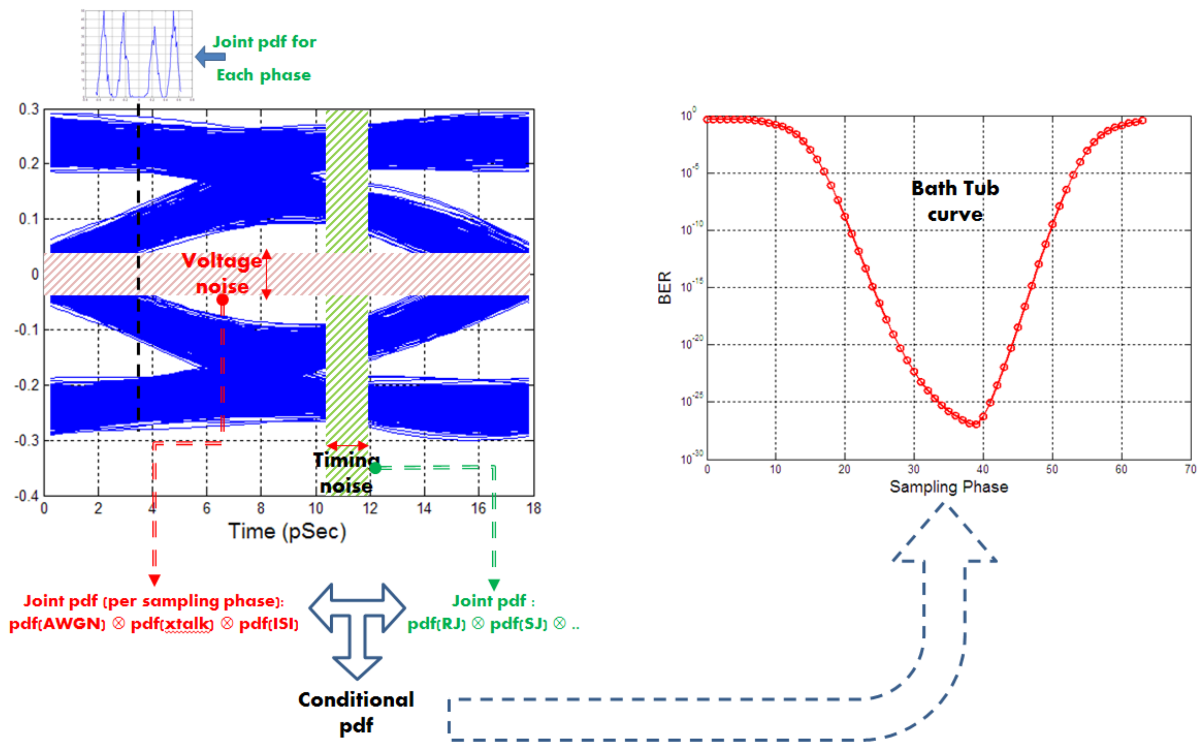


**Figure 2**

## Modeling of Non linearity

One method of modeling nonlinear behavior in a system is to concatenate a linear system with a power series (also called polynomial functions). This has been observed to match closely with actual circuits (such as CTLE in a high speed link). Input (X) and output (Y) of the nonlinear block are related by:

$$Y = \sum_{n} \alpha_n X^n$$

Where $\alpha_n$ are the respective coefficients of $n^{th}$ order nonlinearity. For differential signaling, usually n is odd since even order nonlinearities are absent. Thus, the above expression reduces to:

$$Y = \alpha_1 X^1 + \alpha_3 X^3 + \alpha_5 X^5 + \cdots$$

By definition this type of modeling assumes static nonlinearity. Another more accurate method of modeling frequency dependent non linearity is by using Volterra series. Although the methodology of this work can be extended to modeling frequency dependent non linearity, the results presented in this paper are directly applicable to static non linearity only.

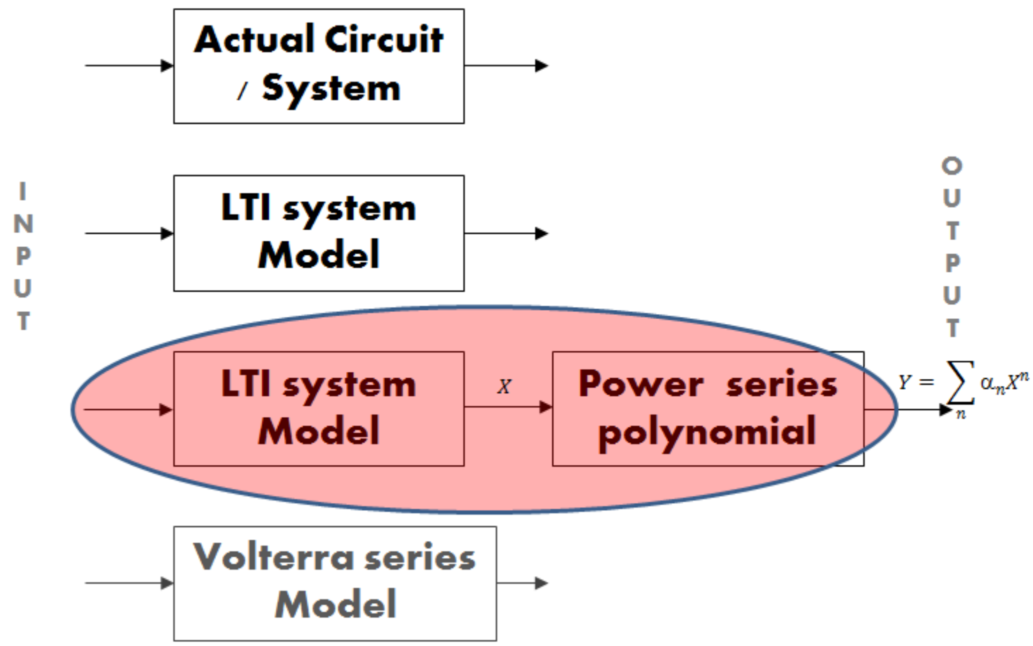


**Figure 3**

Depending upon the type and order of nonlinearity, it may or may not be straightforward to extract the coefficients $\{\alpha_n\}$. Here we describe one method which uses time domain waveforms (broadband signal) at the input and output of the actual circuit to extract the coefficients $\{\alpha_n\}$. Circuit design usually starts form a specification in form of an ideal model (one common example being pole-zero model used to generate CTLE transfer functions). Input to the circuit (say channel output) is also known.

Refer Figure 4 .Let $Y$ denote output of actual circuit (for example this could be an H-spice model), $X$ denote the output of linear model (ideal circuit), and $Y'$ denote the output of nonlinearity block in front of the linear model i.e. $Y' = \sum_n \alpha_n X^n$.

If $[x_1, x_2, \ldots x_k]$ denote the time domain samples of $X$, $[y_1, y_2, \ldots y_k]$ denote the time domain samples of $Y$, then $Y$ can be written in terms of $X$ & $\{\alpha_n\}$ as:

$$\begin{bmatrix} x_1^1 & x_1^2 & & x_1^N \\ \vdots & \vdots & \cdots & \vdots \\ x_k^1 & x_k^2 & & x_k^N \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}$$

Denoting:

$M = \begin{bmatrix} x_1^1 & x_1^2 & & x_1^N \\ \vdots & \vdots & \cdots & \vdots \\ x_k^1 & x_k^2 & & x_k^N \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$ & $X^2 = \begin{bmatrix} x_1^2 \\ \vdots \\ x_k^2 \end{bmatrix}$, the $\{\alpha_n\}$ can then be extracted as:

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = M^{-1} Y$$

Note that this matrix inversion (zero forcing) approach is not optimum in a mean square sense, but it gives a very good estimate of NL terms $\{\alpha_n\}$ as shown in Figure 5.

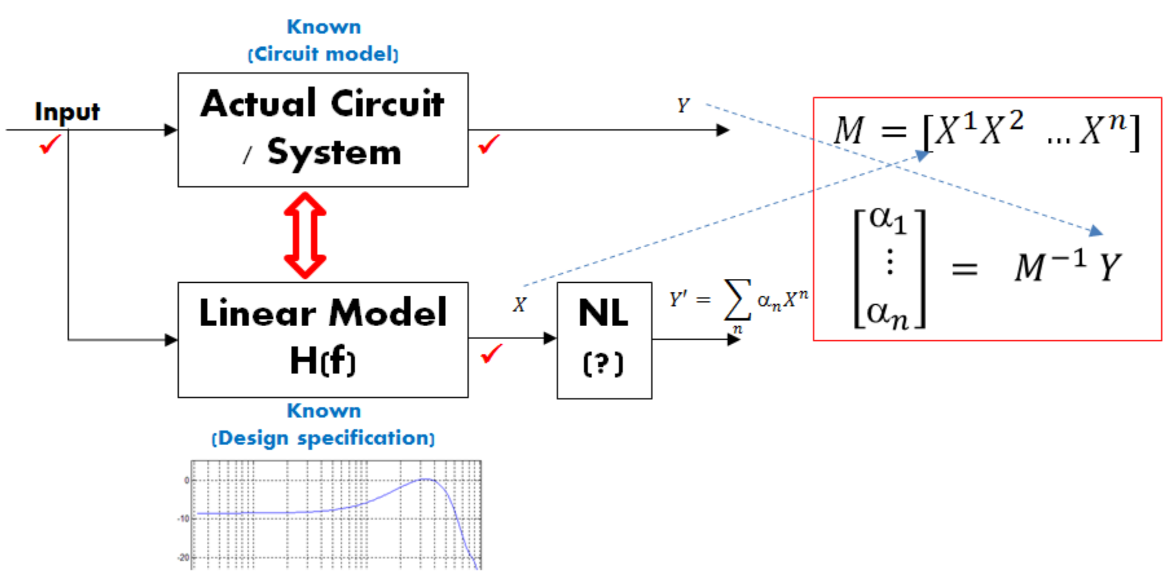


**Figure 4**

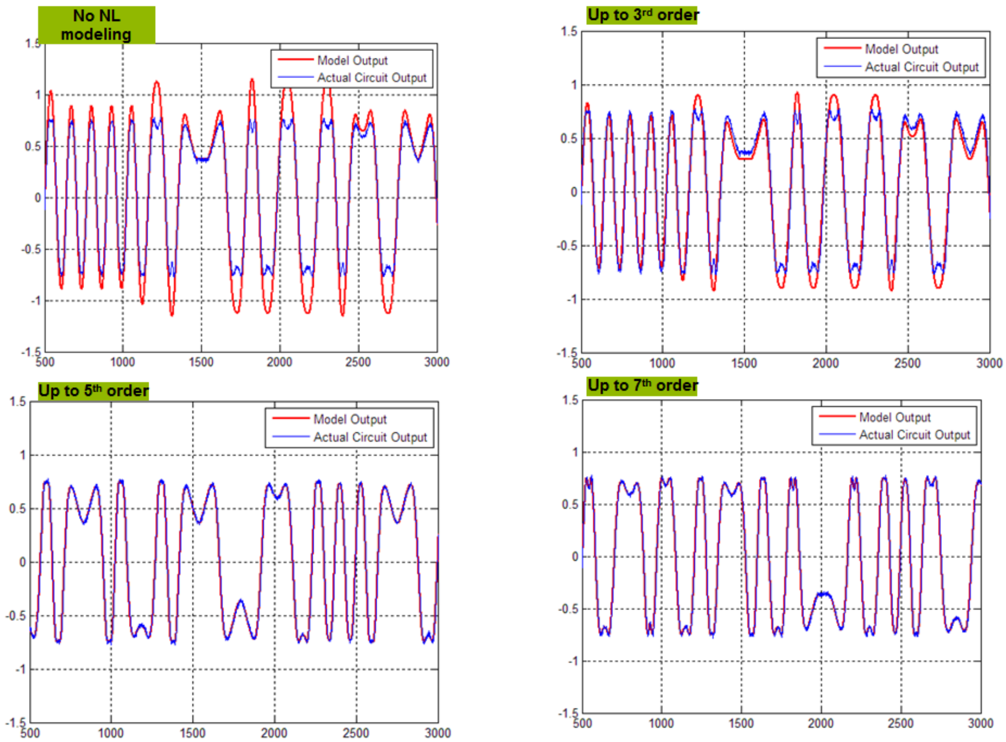


**Figure 5**

Table 1 below quantifies the error in estimation as a function of the number of terms used in NL coefficients. Error is defined as $Y - Y'$, is normalized to signal power and expressed in dB. [Entries of the table are thus a measure of SNR: $\sigma_y^2 / \sigma_{error}^2$ (dB) where 'N' in SNR denotes error power due to non-inclusion of NL].

**Table 1:** $\sigma_y^2 / \sigma_{error}^2$ **(dB). Error =** $Y - Y'$.

| No NL modeling | Up to 3rd order | Up to 5th order | Up to 7th order |
|---|---|---|---|
| 11 dB | 23 dB | 46 dB | 51 dB |

To simplify illustrations, for the rest of the paper we shall consider a simple nonlinearity model of the form $Y = X - 0.3X^3$ in the subsequent sections. The input VS output characteristics of such a system are shown in Figure 6.

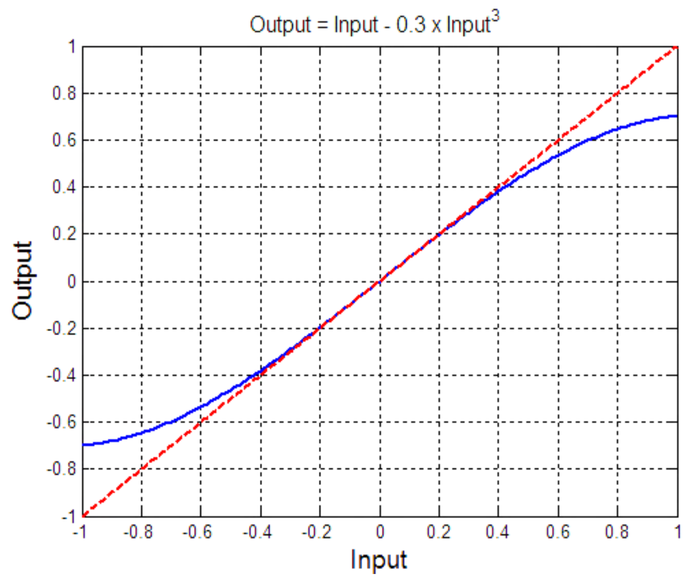


**Figure 6**

## Modification of probability density function in presence on non-linearity

Let the probability density function of a random variable X be given as $F_X(x)$. X can represent the output of a linear system say channel output or CTLE output. Let y = g(x) represent the output of a non-linear function whose input is x. The PDF of Y, $F_Y(y)$ can be determined in terms of pdf of X as: [*Probability, Random variables and Stochastic Processes: Athanasios Papoulis, Section 5-2*]

$$F_Y(y) = \frac{F_X(x_1)}{|g'(x_1)|} + \frac{F_X(x_2)}{|g'(x_2)|} + \cdots \frac{F_X(x_n)}{|g'(x_n)|}$$

Where $x_1$, $x_2$ … $x_n$ are the n roots of y=g(x), i.e. y = $g(x_1) = g(x_2) = \ldots g(x_n)$. g′(x) denotes the derivative of g(x).

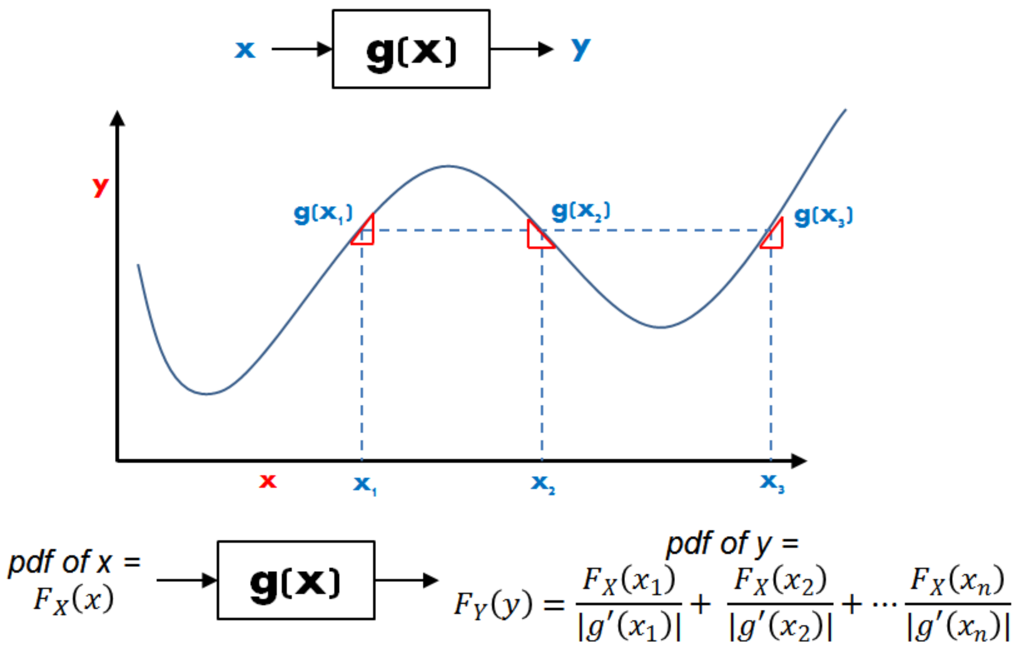


**Figure 7**

For a monotonic function g(X) (which is usually the case in high speed links), $F_Y(y)$ can be simplified as:

$$F_Y(y) \quad = \quad \left|\frac{dx}{dy}\right| F_X(x)$$

The variable Y now represents the output of a non-linear function. To calculate the PDF of a linear system given the PDF of the input and the transfer function of the system is well known. Knowing these 2 techniques, we can concatenate any number of linear system components with any number of non-linear components in any order. Once we know the PDF of the signal and impairments which have passed through a combination of linear & non-linear elements, we can determine the BER using tail probability methods which are also well known.

To demonstrate the concept, first we will apply the above approach to a simple example of signal in presence of AWGN (since the PDF here is Gaussian which has a well-known closed form solution), in the next section we will consider a high speed link with ISI, crosstalk, AWGN and jitter. Figure 8 shows the model for signal in presence of white Gaussian noise (AWGN). Input (X) to the nonlinear block has a well-defined PDF:

PDF of output Y (where $Y = X + \alpha X^3$) can be expressed as:

$$F_Y(y) = \left|\frac{dx}{dy}\right| F_X(x)$$

$$= \left|\frac{1}{1+3\alpha X^2}\right| F_X(x)$$

Where $F_X(x)$ denotes the normal distribution: $\frac{1}{\sigma\sqrt{2\Pi}}\, e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$

Note the plot of the term $\left|\frac{dx}{dy}\right|$. The input PDF is weighted by this term to create the PDF of output Y. For small values of input, this is equal to 1 (unity), denoting no change in PDF. As the input grows in magnitude, the 'companding' effect of non-linearity becomes clear. The large values of input are now going to 'collect' around +/-1. The input PDF thus gets warped in accordance with the term $\left|\frac{dx}{dy}\right|$.

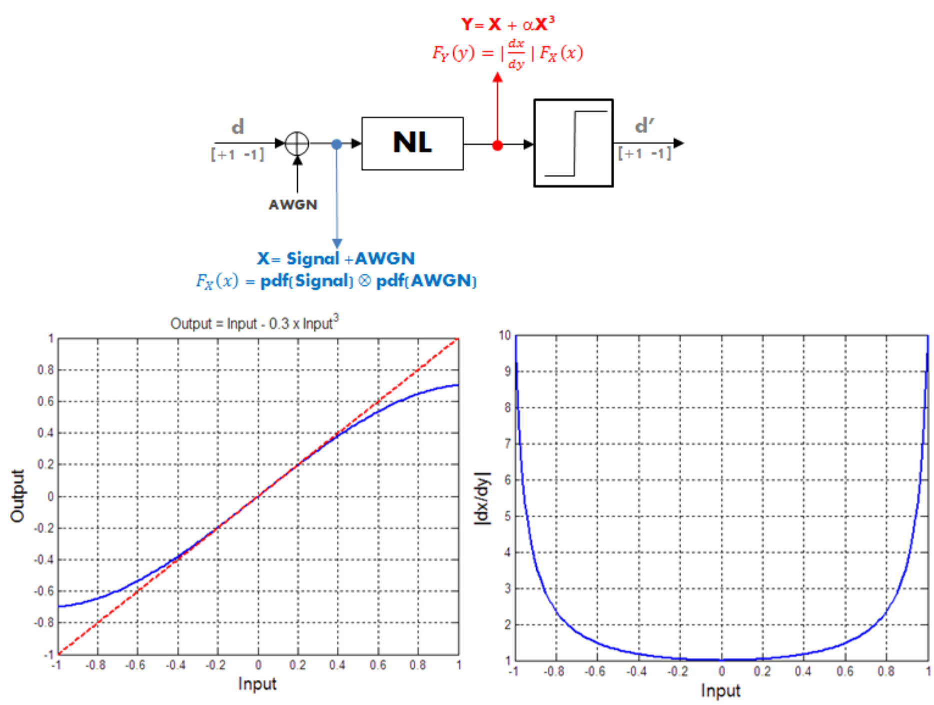


**Figure 8**

To validate that the above formulation is correct, we simulate the AWGN link model for two symbol sets: PAM2 ($d_k$ = [-1 1]) & PAM4 ($d_k$ = [-3 -1 1 3]/3) and compare the PDF obtained via simulation (histogram function in Matlab) VS analytically derived PDF. As shown in Figure 9, there is a perfect match. The objective of considering PAM4 is to show that higher order modulations suffer more from nonlinearity as will become clear in BER results of following section.

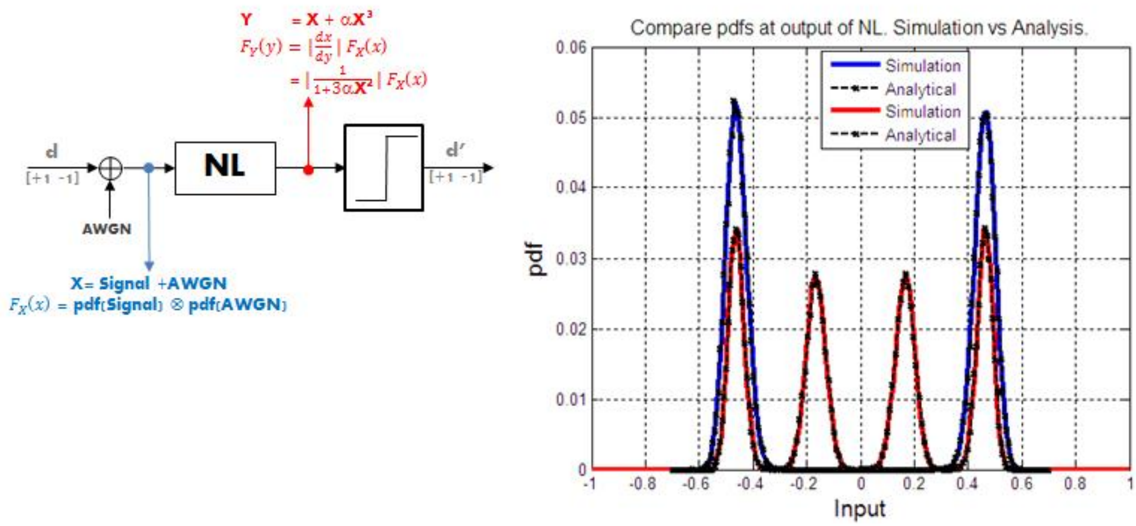


**Figure 9**

In general we should expect higher order modulations to be more sensitive to nonlinearity, since the outer most points in the constellation are affected the most and they will dominate the BER. This fact is intuitively clear in Figure 10. It is conceivable that the detection rule may be modified to take advantage of (known) non-linearity. This paper assumes that same detection rule (minimum distance) as is used for linear system analysis is used for calculating BER in presence of non-linearity.

Figure 13 shows the BER results for NRZ signaling.

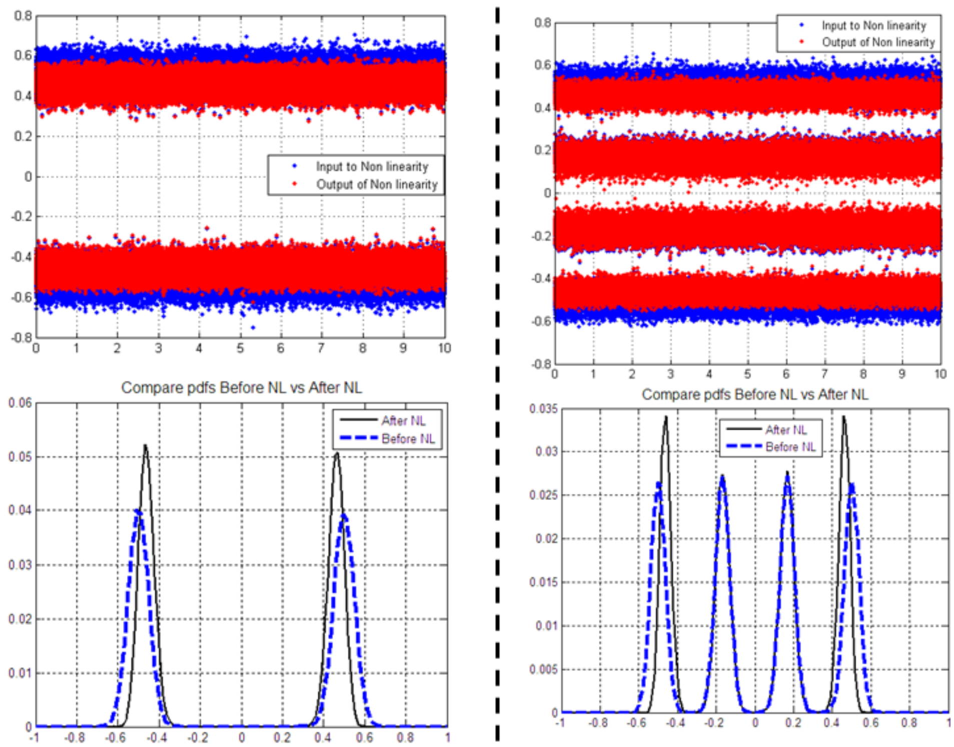


**Figure 10**

## Results for a typical high speed link

In addition to background noise, a typical high speed link has a dispersive channel creating inter symbol interference (ISI), crosstalk from neighboring links and timing jitter on transmit and receive sampling clocks. Then net signal also goes through an analog front end (typically a CTLE) which is designed to equalize the ISI effects of the dispersive channel. The AFE compensates for ISI but also inevitably boosts noise and creates nonlinear behavior.

Figure 11 shows a simplified link model for a typical high speed transceiver. For the purpose of analysis we will combine the linear segments of Transmit, package, connector, through channel, Analog frond end (CTLE) into an effective 'Channel' as shown on the bottom of Figure 11. The channel before CTLE has a 20dB loss at Nyquist frequency. (Note that for PAM4 signaling, we use only half the bandwidth of PAM2 for both through & crosstalk). Two components of jitter are considered – RJ (random jitter) with a Gaussian PDF and duty cycle distortion (DCD).

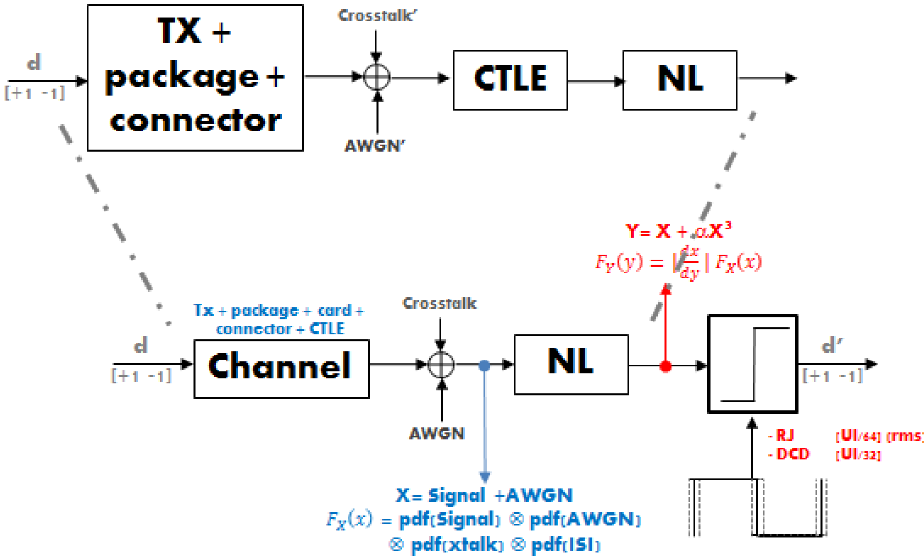


**Figure 11**

Figure 12 shows the frequency domain characteristics of the insertion loss and crosstalk channel along with the frequency boosting of CTLE. A rough measure of signal quality can be obtained via eye diagram at the decision slicer input. The blue curves denote the eye diagram before NL, the red curves are after the signal passes though NL.

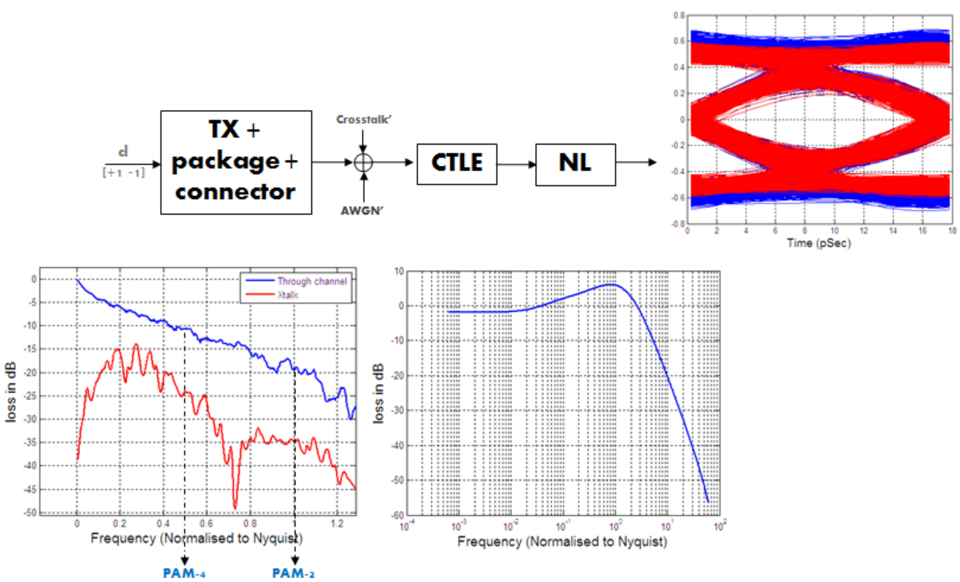


**Figure 12**

Since the objective here is to evaluate the effects of NL, to make a fair comparison between PAM2 & PAM4, we should start from the same baseline, i.e. same BER without NL. The bandwidths of insertion loss, crosstalk, AWGN and CTLE for PAM2 are half that of PAM4. Jitter is specified as a fraction of UI, so that automatically adjusts for signaling rate. Since the crosstalk channel is not flat, we have to make small adjustment on gain of crosstalk channel to make the baseline BER (without NL) the same for both PAM2 & PAM4.

It is conceivable that the detection rule may be modified to take advantage of (known) non-linearity. This paper assumes that same detection rule (minimum distance) as is used for linear system analysis is used for calculating BER in presence of non-linearity. Figure 13 shows the BER results for NRZ signaling.

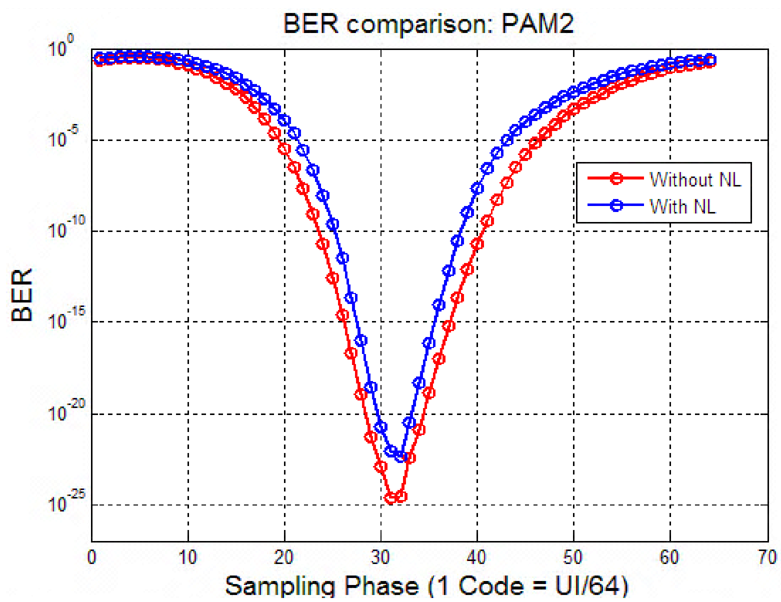


**Figure 13**

Figure 14 shows the BER results for PAM4 signaling.1 UI for PAM4 is twice as wide in time than PAM2.

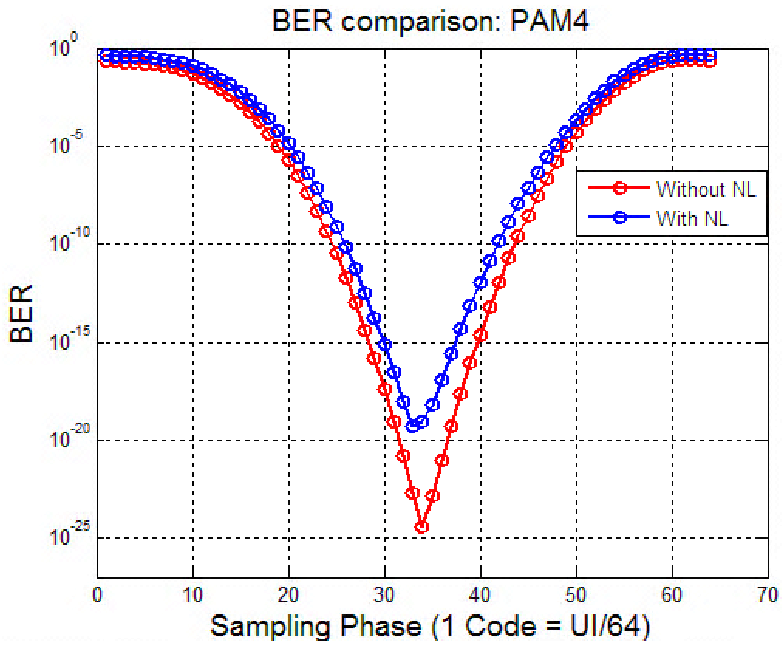


**Figure 14**

Note that the baseline BER (without NL) is roughly the same for both PAM2 & PAM4 (by design), but the final BER after NL is significantly worse for PAM4 than PAM2. Table 2 compares the two modulation schemes. BER is compared for the best sampling phase.

**Table 2**

| | Bandwidth (Nyquist) | UI | BER without NL | BER with NL |
|---|---|---|---|---|
| PAM2 | $F_N$ | UI_PAM2 = 1/(2* $F_N$) | 1e-25 | 1e-23 |
| PAM4 | $F_N$ / 2 | 2*UI_PAM2 | 1e-25 | 1e-20 |

## Link model with multiple linear and nonlinear blocks

Here we illustrate how to transform PDFs in a link with multiple linear and nonlinear blocks concatenated with each other. Basically PDF at the output of linear block can obtained via well-known convolution method [Input PDF is weighted by the coefficients of the impulse response and convolved with appropriate matching of x-axis]. For the PDF at the output a NL block, we can use the methodology developed in this paper, i.e.

$$F_Y(y) = \frac{F_X(x_1)}{|g\prime(x_1)|} + \frac{F_X(x_2)}{|g\prime(x_2)|} + \cdots \frac{F_X(x_n)}{|g\prime(x_n)|}.$$

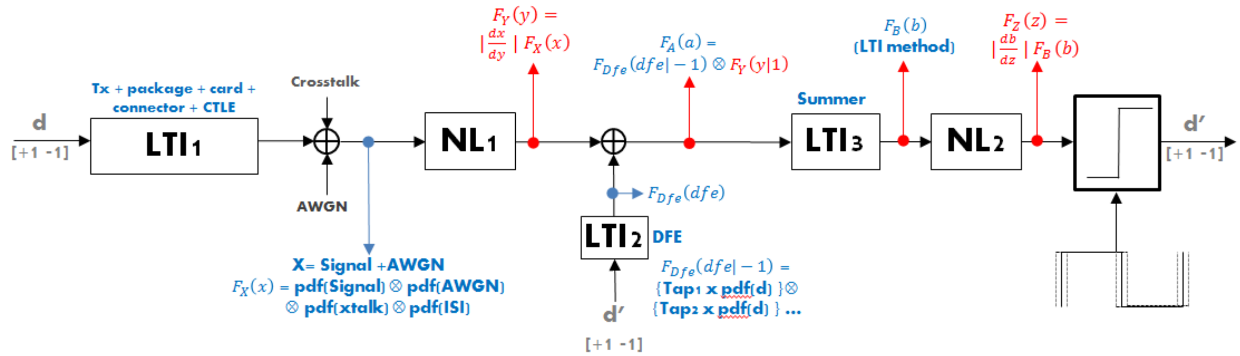


**Figure 15**

## Summary

We presented a method for calculating bit error rate (BER) using probability density function (PDF) of impairments (ISI, crosstalk, jitter etc.) in presence of nonlinearity. The method was based on determining the PDF at the output of nonlinearity given the PDF at its input. Once the PDF is correctly modified, the tail probability methods to determine BER can be applied. This way we can combine any linear system(s) with a non-linear system(s) and predict its BER based on knowledge of channel and impairments. Static, memory-less non-linearity polynomials were considered and results presented for typical high speed links running NRZ and PAM4 modulations. The methodology can also be adapted for time varying, frequency dependent nonlinearity, the work for adopting the presented approach for frequency dependent nonlinearity is going on.

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